

Studies of the Schroedinger-Newton Equations in D Dimensions

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Abstract

We investigate a D dimensional generalization of the Schroedinger-Newton equations, which purport to describe quantum state reduction as resulting from gravitational effects. For a single particle, the system is a combination of the Schroedinger and Poisson equations modified so that the probability density of the wavefunction is the source of the potential in the Schroedinger equation. For spherically symmetric wavefunctions, a discrete set of energy eigenvalue solutions emerges for dimensions $D < 6$, accumulating at $D = 6$. Invoking Heisenberg's uncertainty principle to assign timescales of collapse corresponding to each energy eigenvalue, we find that these timescales may vary by many orders of magnitude depending on dimension. For example, the time taken for the wavefunction of a free neutron in a spherically symmetric state to collapse is many orders of magnitude longer than the age of the universe, whereas for one confined to a box of picometer-sized cross-sectional dimensions the collapse time is about two weeks.

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1 Introduction

In quantum mechanics, objects are described by wavefunctions. These take the form of complex superpositions of various evolutionary alternatives, or states. Although successful in describing many aspects of the quantum world, this picture often leads to troubling interpretations when extrapolated to the macroscopic level. One issue that has suffered long debate is the fact that one never observes a superposition of states. Rather, one only observes a system's *basic* or *stationary* states. We are therefore forced to provide a mechanism by which quantum wavefunctions reduce to their stationary states. This process is called *wavefunction collapse* or *state reduction*. Motivated by the basic conflicts which exist between general relativity and quantum mechanics, a number of authors have proposed the idea that wavefunction collapse is an objective phenomenon which arises due to gravitational effects [1]. For example Penrose [2] has suggested a scheme in which a superposition of two stationary quantum states should be fundamentally unstable if there exists a significant mass displacement between them. In this case there should be some characteristic timescale T_G for decay into the basic states. Although a detailed estimate of T_G would require a full theory of quantum gravity, under this hypothesis it is reasonable to expect that for non-relativistic systems

$$T_G \simeq \frac{\hbar}{\Delta E_G} \quad (1)$$

where ΔE_G is the gravitational self-energy of the difference between the mass distributions of the two states.

The explicit nature of the basic states in this consideration is somewhat unclear. We cannot simply regard the position of a lump of mass as a basic state, because then we would be forced to regard any general state of a particle as a superposition. As a possible solution to this problem, Penrose proposes that these (non-relativistic) basic states are solutions of the Schroedinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\Psi + U\Psi = E\Psi \quad (2)$$

where the additional term represents a coupling to a certain gravitational potential U . This potential is determined (via the Poisson equation) by the expectation value of the mass distribution in the state determined by the wavefunction. For single particle systems, the matter density is determined by the probability density from the wavefunction, and so

$$\nabla^2 U = 4\pi G m^2 |\Psi|^2 \quad (3)$$

where G is Newton's gravitational constant, and m is the mass of the single particle. Equations (2,3) are dubbed the Schroedinger-Newton (SN) equations [3]. A preliminary investigation of the properties of the solutions to the SN equations was recently carried out by Moroz et. al. [4]. Under the assumptions of spherical symmetry in 3 dimensions, and by demanding only that U and Ψ be

everywhere smooth, they discovered a discrete family of bound state solutions, labelled by an integer $n \geq 0$. Each solution is a normalizable wavefunction, and the n th solution has n zeros. The energy eigenvalues associated with each of these solutions are negative, and monotonically converge to zero for large n . These results can be justified analytically [5]. The energy eigenvalues are the differences between a given bound state and a continuum ‘superposition’ state, and so provide via (1) an estimate of the timescale of self-collapse of a single particle of mass m . The energy eigenvalues scale like m^5 , and so particles of small mass have extremely long self-collapse times – for a nucleon mass the estimate is 10^{53} s [4]. A recent related study by Soni is commensurate with these results [6].

Relaxing the assumption of spherical symmetry is in general a difficult task due to the non-linearity of the SN equations. However there are two situations in which this is fairly straightforward: cylindrical symmetry with no angular momentum and planar symmetry. Rewriting the SN equations for these cases effectively reduces them to 2 and 1 dimensional situations respectively. These cases, along with the spherically symmetric case, can be simultaneously recovered by rewriting the spherically symmetric SN equations in D dimensions. Motivated by the above, we consider in this paper an analysis of the D -dimensional spherically symmetric SN equations, for $D \geq 1$. Although the higher-dimensional cases are of less direct physical interest than the $D = 2, 3$ cases, such a study affords us some insight into the dimensional behaviour of the SN system. This behaviour may be of more than pure pedagogical interest since many candidate approaches to quantum gravity are typically cast in higher dimensions (superstring theory being the obvious example).

2 The D-dimensional SN equations

Any solution to the SN equations (2,3) must be normalizable (i.e. square-integrable). Integrating the probability density over all space yields

$$k^2 = \int_0^\infty d^D x |\Psi|^2 \quad (4)$$

where k is a dimensionless number, and so the wavefunction must be rescaled to ensure there is unit probability of finding the particle somewhere in space. Writing $\Psi = k\psi$, the SN equations then become

$$-\frac{\hbar^2}{2m} \nabla^2 \psi + U\psi = E\psi \quad (5)$$

$$\nabla^2 U = 4\pi G k^2 m^2 |\psi|^2 \quad (6)$$

$$\int_0^\infty d^D x |\psi|^2 = 1 \quad (7)$$

and we see that the normalization factor enters the system due to its non-linearity. Redefining variables in (5,6) via [4]

$$\psi = \alpha \mathcal{S} \quad E - U = \beta \mathcal{V} \quad (8)$$

where

$$\alpha = \frac{\hbar}{\sqrt{8\pi G k^2 m^3}} = \frac{\hat{\alpha}}{k} \quad \beta = \frac{\hbar^2}{2m} \quad (9)$$

yields

$$\nabla^2 \mathcal{S} = -\mathcal{S} \mathcal{V} \quad (10)$$

$$\nabla^2 \mathcal{V} = -\mathcal{S}^2 \quad (11)$$

$$\alpha^2 \int_0^\infty d^D x \mathcal{S}^2 = 1 \quad (12)$$

where we can assume that ψ is real without loss of generality. The parameters $\hat{\alpha}$ and β have units of $(\text{length})^{2-D/2}$ and $(\text{length})^2 \times \text{energy}$ respectively. The system (10,11) is invariant under the rescaling transformation

$$(\mathcal{S}, \mathcal{V}, x) \longrightarrow (\lambda^2 \mathcal{S}, \lambda^2 \mathcal{V}, \lambda^{-1} x) \quad (13)$$

independent of the dimension D , where λ has units of inverse length. Using this transformation we can rewrite the system in terms of fully dimensionless functions (S, V) of dimensionless variables. For the spherically symmetric case the D -dimensional Laplacian operator is $\nabla^2 \varphi = r^{1-D} (r^{D-1} \varphi')'$ and so (10,11,12) become

$$(r^{D-1} S')' = -r^{D-1} S V \quad (14)$$

$$(r^{D-1} V')' = -r^{D-1} S^2 \quad (15)$$

$$\alpha^2 \lambda^{4-D} \int_0^\infty d^D x S^2 = 1 \quad (16)$$

where the prime denotes differentiation with respect to r . We must require that the integral in (16) be finite in order for the SN system to be physically meaningful. We therefore seek solutions to the SN system that are finite for all values of $r \geq 0$; continuity requirements imply that these solutions are everywhere smooth. Note that $\alpha_0^2 \lambda^{4-D}$ is a dimensionless quantity.

However, our equation (16) clearly has a problem for $D = 4$, where we lose our ability to rescale using λ . Because we still require the wavefunction to be normalized to unity, we are forced to introduce a constant $\mathfrak{K} > 0$ in the $D = 4$ case so that $S(r) \rightarrow \sqrt{\mathfrak{K}} S(r)$. Then our equations (14), (15), and (16) become

$$(r^3 S')' = -r^3 S V \quad (17)$$

$$(r^3 V')' = -\mathfrak{K} r^3 S^2 \quad (18)$$

$$\alpha^2 \mathfrak{K} \int_0^\infty d^4 x S^2 = 1 \quad (19)$$

We will not always include this constant \mathfrak{K} explicitly in the following discussion. It will simply be understood to occur in the rescaling when $D = 4$. Smoothness implies that $S'(0)$ and $V'(0)$ are finite. With this information we can rewrite (14,15) as

$$S(r) = S_0 + \frac{1}{2-D} \int_0^r x \left(1 - \left(\frac{x}{r}\right)^{D-2}\right) S(x)V(x)dx \quad (20)$$

$$V(r) = V_0 + \frac{1}{2-D} \int_0^r x \left(1 - \left(\frac{x}{r}\right)^{D-2}\right) S^2(x)dx \quad (21)$$

It is straightforward to show that Picard's theorem [7] is satisfied by this system of equations, and so given S_0 and V_0 a unique solution to (14,15) exists within a range $[0, R(S_0, V_0)]$. The $D < 3$ versions of the integral equations (20,21) require some care. Integrating (14,15) explicitly for these cases yields

$$S(r) = S_0 + \int_0^r (x-r) S(x)V(x)dx \quad (22)$$

$$V(r) = V_0 + \int_0^r (x-r) S^2(x)dx \quad (23)$$

for $D = 1$ and

$$S(r) = S_0 + \int_0^r x \ln\left(\frac{x}{r}\right) S(x)V(x)dx \quad (24)$$

$$V(r) = V_0 + \int_0^r x \ln\left(\frac{x}{r}\right) S^2(x)dx \quad (25)$$

for $D = 2$. Both of these sets of equations may be obtained formally from (20,21) by inserting these values of D , the latter case being understood as the limit $D \rightarrow 2$.

Note that for all of these cases the function $V(r)$ is monotonically decreasing since

$$V'(r) = - \int_0^r \left(\frac{x}{r}\right)^{D-1} S^2(x)dx \quad (26)$$

a result valid for all $D \geq 1$. Hence if $V_0 \leq 0$, then $\lim_{r \rightarrow \infty} V(r) = -\infty$ and so $S(r)$ will also diverge for large r . Consequently only $V_0 > 0$ is of physical interest. For $V_0 > 0$ the function $V(r) > 0$ initially. By rewriting $S(r) = r^{\frac{2-D}{2}} s(r)$ equation (14) can be rewritten in the form

$$r^2 s'' + r s' + \left[r^2 V(r) - \left(\frac{D-2}{2}\right)^2 \right] s = 0 \quad (27)$$

which for $V(r) = V_0$ is Bessel's equation. Hence near the origin we expect $s(r)$ to have oscillatory behaviour. However when $V(r)$ becomes negative eq. (27) is similar to the modified Bessel equation and the behaviour of $s(r)$ will be a linear

combination of exponentially amplified and damped functions. Normalization requirements imply that only the exponentially damped solutions are allowed. In this case $S(r)$ decays like an exponential times an inverse power of r and so the integral in (21) will be finite, yielding a finite $V(r)$ at large r . This behaviour was already noted for the $D = 3$ case [4]; we see here that it is valid for arbitrary $D > 0$.

3 Numerical Study of the D-dimensional SN Equations

The form (14,15) of the SN equations is not well-suited for numerical study. A more convenient form is

$$(rS)'' = -rSV + (3-D)S' \quad (28)$$

$$(rV)'' = -rS^2 + (3-D)V' \quad (29)$$

We are interested in finding solutions to the system (28,29) for that are smooth and finite for all r . These are the bound state solutions referred to earlier. We will require our initial conditions, $S(0) = S_0$ and $V(0) = V_0$ to be greater than zero, a constraint which avoids both the trivial solutions $S = V = 0$ and the non-normalizable solutions $V = \pm S = -2(D-4)r^{-2}$, and is consistent with the rescaling freedom (13). It is straightforward to obtain the power-series solutions to (28,29) near the origin:

$$S(r) = S_0 - \frac{S_0 V_0}{2D} r^2 + \frac{S_0 (S_0^2 + V_0^2)}{8D(D+2)} r^4 - \frac{S_0 V_0 ((5D+4)S_0^2 + DV_0^2)}{48D^2(D+2)(D+4)} r^6 + \dots \quad (30)$$

$$V(r) = V_0 - \frac{S_0^2}{2D} r^2 + \frac{S_0^2 V_0}{4D(D+2)} r^4 - \frac{S_0^2 ((2D+2)V_0^2 + DS_0^2)}{24D^2(D+2)(D+4)} r^6 + \dots \quad (31)$$

where the functions S and V and their first derivatives are required to be finite at the origin. Using the rescaling freedom to set $V_0 = 1$ (which is equivalent to setting $\lambda^2 = V_0$), we see that for any dimension D the solutions near the origin depend only on a single free parameter.

Integrating the system (28,29) using a Fehlberg fourth-fifth order Runge-Kutta method in MAPLE we find that in each dimension D an infinite set of discrete bound states appears as expected. These can be identified by the number of local extrema. An infinite amount of precision is required in the choice of S_0 such that the solutions do not diverge as r increases. This value of S_0 marks the transition between solutions in which $S(r)$ diverges to $+\infty$ and solutions where $S(r)$ diverges to $-\infty$. As we increase the accuracy of S_0 for a specific bound state, the value of r at which $S(r)$ blows up increases. We will see that the values for S_0 and the distance between bound states vary significantly with dimension.

The general method for finding solutions to the D -dimensional system is the same as that in the $D = 3$ case [4]. Choose a value of $S_0 = S_0^{(n+1)+}$

for which $S(r)$ has n zeros and diverges to $(-1)^n \infty$ at finite r . Then select a slightly smaller value of $S_0 = S_0^{(n+1)-}$ which has $n+1$ zeros and diverges to $(-1)^{n+1} \infty$ for some finite r . Between these two values of S_0 is a value $S_0^{(n+1)-} < \hat{S}_0^{(n+1)} < S_0^{(n+1)+}$ for which $S(r)$ has n zeros and is smooth and finite for all values of r and is square-integrable. The potential $V(r)$ will also be smooth and finite, with finite energy eigenvalue E_{n+1} . By successively narrowing this interval the bound-state value of S_0 can be achieved to any desired accuracy. The bound state wavefunction with $n+1$ zeros may be obtained by choosing another value $S_0^{(n+2)+} < S_0^{(n+1)-}$ and repeating the procedure with $n \rightarrow n+1$. We find that the values $\hat{S}_0^{(n+1)}$ are rapidly decreasing functions of the dimension D , and that the bound state solutions appear to accumulate at $D=6$. We are limited by numerical accuracy in carrying out our investigations for dimensions $D \geq 6$ due to this effect. A summary of this data is presented in Table 1.

| D | \hat{S}_0^1 | \hat{S}_0^2 | \hat{S}_0^3 | \hat{S}_0^4 |
|-----|---------------|---------------|---------------|---------------|
| 1 | 1.5583977884 | 0.379904201 | 0.2128374651 | 0.1475990005 |
| 2 | 1.2134344293 | 0.6482524055 | 0.4937184140 | 0.41447908 |
| 3 | 1.0886370794 | 0.8264742841 | 0.7442133785 | 0.70014479 |
| 4 | 1.0327684253 | 0.9309542414 | 0.9053504436 | 0.89422924 |
| 5 | 1.0081005592 | 0.9825730584 | 0.9795819080 | > 0.978 |
| 6 | 1.0000000000 | 1.0000000000 | 1.0000000000 | 1 |

Table 1. Initial conditions $\hat{S}_0^{(n+1)}$ for the first ($n=0$) to fourth ($n=3$) bound state wavefunctions, for one to six spatial dimensions. The accumulation of the higher bound states for large dimensions results in the decreasing accuracy of the lower right hand entries of the table.

Figures 1,2 and 3 illustrate the solutions for S for $D=1$, for the first, second and third bound state transitions. Figures 4 to 8 illustrate the first bound state transitions for dimensions $D=2$ to $D=6$. Figure 9 illustrates the accumulation effect at $D=6$.

We turn now to the problem of determining the energy eigenvalues for the bound states. From (15) and (21) we find that the large- r expansion of $\mathcal{V}(r)$ is

$$\mathcal{V}(r) = \mathcal{A}_D + \frac{\mathcal{B}_D}{D-2} \frac{1}{r^{D-2}} + \dots \quad (32)$$

where

$$\mathcal{A}_D = \mathcal{V}_0 + \frac{1}{2-D} \int_0^\infty x \mathcal{S}^2(x) dx \quad (33)$$

$$\mathcal{B}_D = \int_0^\infty x^{D-1} \mathcal{S}^2(x) dx \quad (34)$$

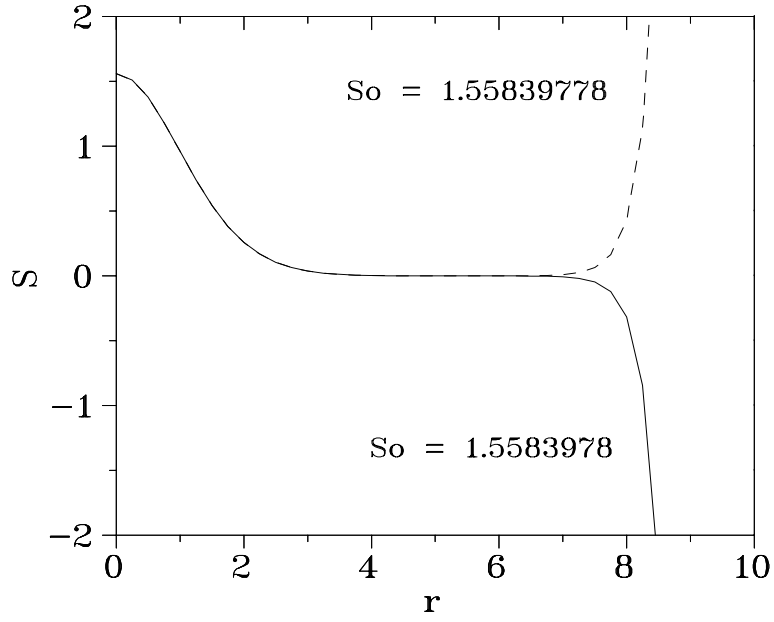


Figure 1: First bound state transition for the 1D SN equations

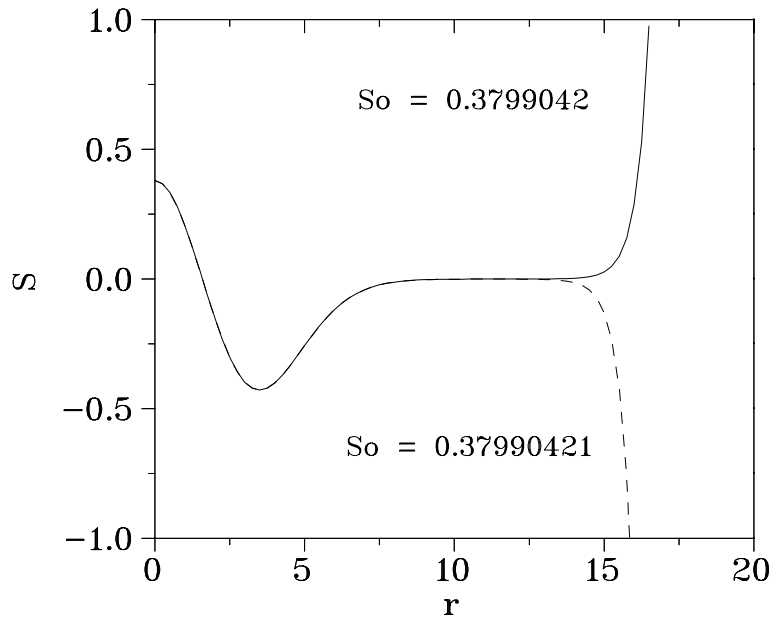


Figure 2: Second bound state transition for the 1D SN equations

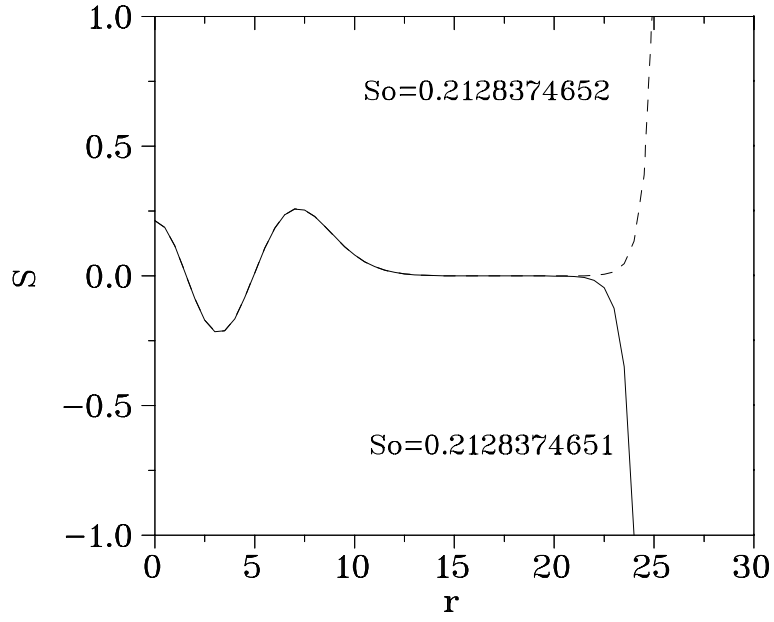


Figure 3: Third bound state transition for the 1D SN equations

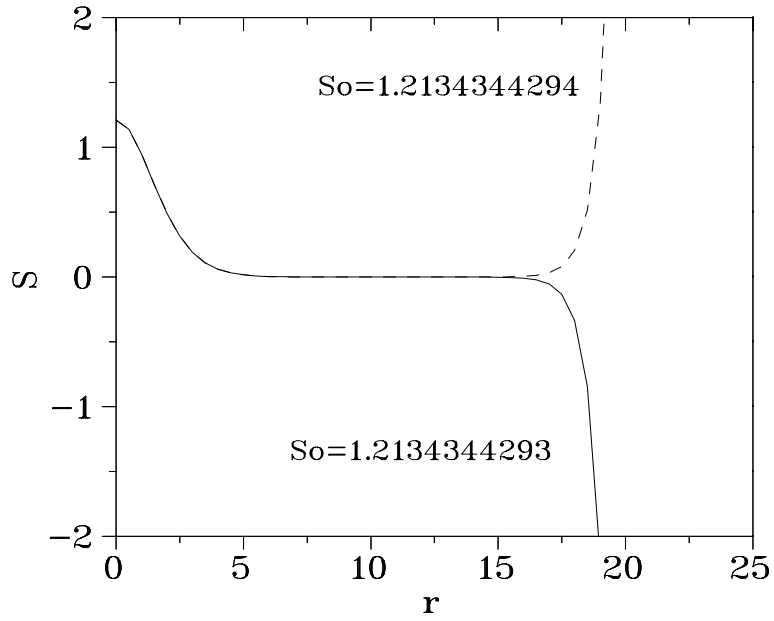


Figure 4: First bound state transition for the 2D SN equations

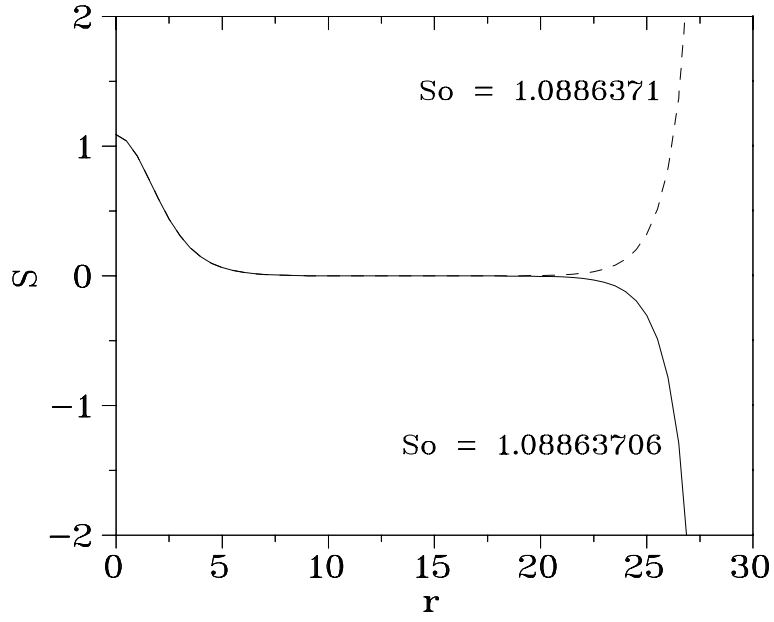


Figure 5: First bound state transition for the 3D SN equations

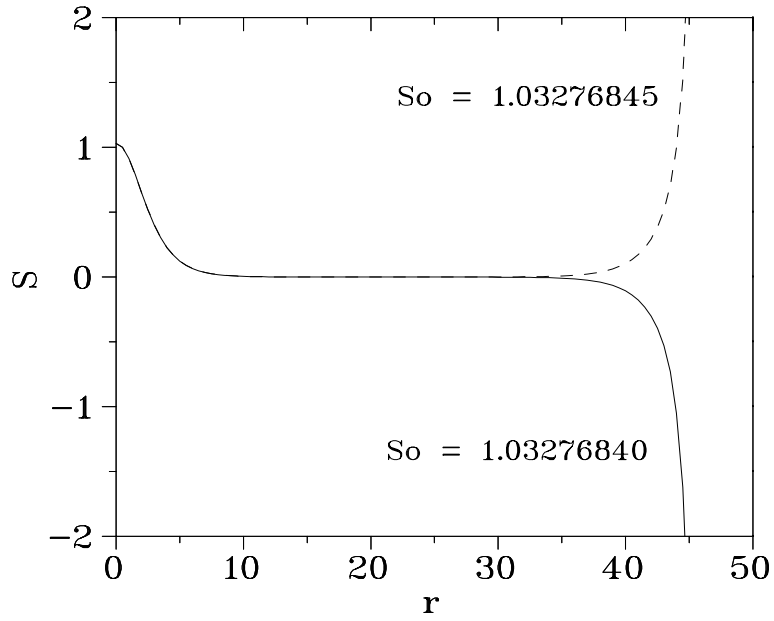


Figure 6: First bound state transition for the 4D SN equations

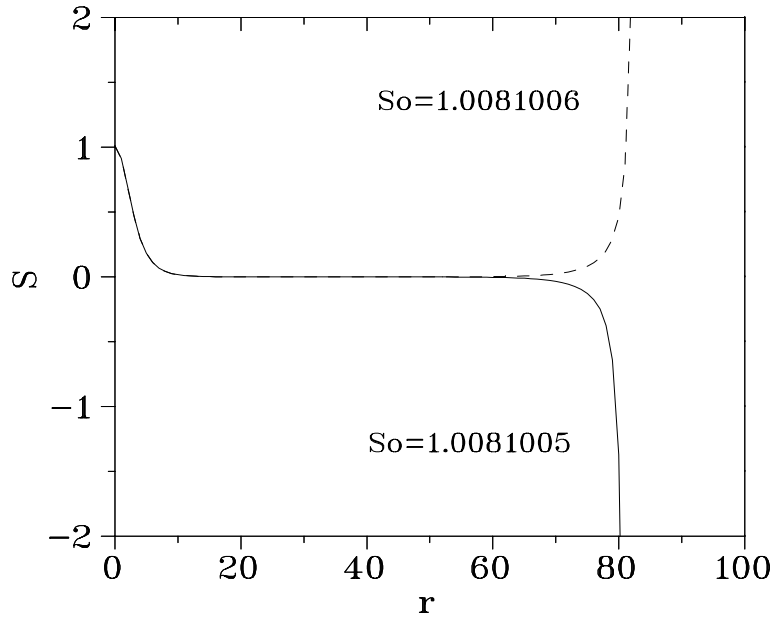


Figure 7: First bound state transition for the 5D SN equations

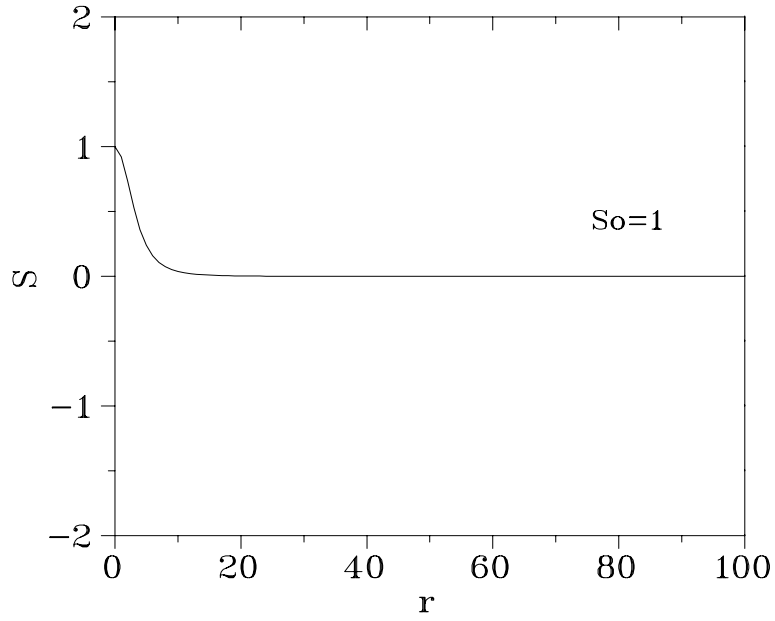


Figure 8: Integration of the 6D SN equations. Bound state characteristics at this and higher dimensions are not apparent.

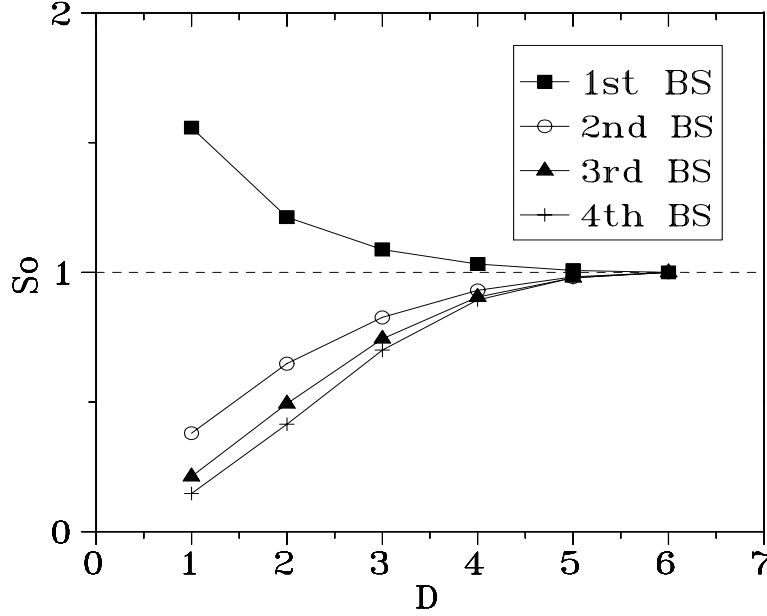


Figure 9: $S(0)$ versus dimension. Bound states appear to accumulate at $D = 6$.

These expansions are formally valid for all $D > 0$; the explicit expansions for $D < 3$ are

$$\mathcal{V}(r) = \begin{cases} \mathcal{A}_1 - \mathcal{B}_1 r + \dots & D = 1 \\ \mathcal{A}_2 - \mathcal{B}_2 \ln\left(\frac{r}{r_0}\right) + \dots & D = 2 \end{cases} \quad (35)$$

and

$$\mathcal{A}_D = \begin{cases} \mathcal{V}_0 + \int_0^\infty x \mathcal{S}^2(x) dx & D = 1 \\ \mathcal{V}_0 + \int_0^\infty x \ln\left(\frac{x}{r_0}\right) \mathcal{S}^2(x) dx & D = 2 \end{cases} \quad (36)$$

$$\mathcal{B}_D = \begin{cases} \int_0^\infty \mathcal{S}^2(x) dx & D = 1 \\ \int_0^\infty x \mathcal{S}^2(x) dx & D = 2 \end{cases} \quad (37)$$

where r_0 is an arbitrary length scale. The energy eigenvalue is given by

$$E_D = \beta \mathcal{V}(r_D) = \beta \mathcal{A}_D \quad (38)$$

The quantity r_D is the point at which the potential $U(r)$ vanishes. For $D \geq 3$, $r_D = \infty$. However for $D = 1, 2$ the potential diverges there and the situation is more delicate. For $D = 1$ a Newtonian gravitational potential diverges linearly with $r = |x|$. Here we extract the r -independent term from (35) to obtain $E_{D=1}$; the result is still given by the right-hand side of (42) with $D = 1$. Effectively we have set $r_D = 0$ in (35) even though this equation is a large r

expansion. For $D = 2$ the Newtonian potential diverges at both large and small r . The normalization point $r_D = r_0$ is therefore arbitrary, and we set $E_{D=1} = \beta \lambda_{D=2}^2 A_{D=2}$ so that (42) remains valid in this case as well. This is tantamount to setting $U(r_0) = \int_{r_0}^{\infty} \times \ln\left(\frac{x}{r_0}\right) S^2(x) dx$. A natural normalization point is to choose r_0 to be the point at which the potential $V(r_0) = 0$. Note

that under the rescaling transformation (13) $(\mathcal{A}, \mathcal{B}) \longrightarrow (\lambda^2 A, \lambda^{4-D} B)$, where

$$A_D = 1 + \frac{1}{2-D} \int_0^{\infty} x S^2(x) dx \quad (39)$$

$$B_D = \int_0^{\infty} x^{D-1} S^2(x) dx \quad (40)$$

and so the combination $\mathcal{A}_D^{4-D}/\mathcal{B}_D^2 = A_D^{4-D}/B_D^2$ is invariant. The parameter λ (or alternatively V_0) is completely arbitrary, serving the sole function of setting the length scales of the problem in units of $\alpha_0^{2/(4-D)}$. Solving (16) for λ yields

$$\lambda_D = \left(\frac{\Gamma\left(\frac{D}{2}\right)}{2\pi^{\frac{D}{2}} \alpha^2 B_D} \right)^{\frac{1}{4-D}} \quad (41)$$

With this normalization, we find that

$$E_D = \beta \mathcal{A}_D = \beta \lambda_D^2 A_D = \beta \left(\frac{\Gamma\left(\frac{D}{2}\right)}{2\pi^{\frac{D}{2}} \alpha^2} \right)^{\frac{2}{4-D}} \frac{A_D}{(B_D)^{\frac{2}{4-D}}} \quad (42)$$

and so the D -dimensional energy is given in units of $\beta/\alpha^{4/(4-D)}$. The quantities A_D and B_D are straightforwardly solved numerically from (33,34). The preceding expression can be rewritten as

$$E_D = \frac{1}{2} m_{pl} c^2 \left(\frac{m}{m_{pl}} \right)^{\frac{D+2}{4-D}} \left(\frac{4\Gamma\left(\frac{D}{2}\right) k^2}{\pi^{\frac{D-2}{2}}} \right)^{\frac{2}{4-D}} \frac{A_D}{(B_D)^{\frac{2}{4-D}}} \quad (43)$$

where $m_{pl} = (\hbar^{D-2}/G_D c^{D-4})^{1/(D-1)}$ is the Planck mass in $D > 1$ dimensions (in one spatial dimension the quantity $c^3/\hbar G_D$ is unitless, and can be absorbed into the normalization constant k).

For $D \leq 3$ the energy eigenvalue is an increasing function of the particle mass, whereas for $D \geq 5$ it is a decreasing function. The most rapid increase is $E_D \sim m^5$ for $D = 3$, with only linear and quadratic dependence in $D = 1$ and 2 respectively. The time-scale for collapse of the state vector is therefore most rapid in a world with three spatial dimensions for any bodies whose mass is appreciably larger than the Planck mass. In a world of more than four dimensions the collapse is fastest for the lightest-mass particles, leading to behaviour which is at complete odds with that expected in the macroscopic world.

If we interpret the lower-dimensional cases to be situations in three spatial dimensions with planar or cylindrical symmetry, a somewhat different picture emerges. Inserting the constants into (43), the dependence of the energy eigenvalues for the planar, cylindrical and spherically symmetric cases are respectively

$$E_p = \frac{(4\pi)^{2/3}}{2} m_{pl} c^2 \left(\frac{m}{m_{pl}} \right) \left(\frac{\ell_{pl}}{\ell} \right)^{\frac{4}{3}} \frac{A_1}{(B_1)^{\frac{2}{3}}} \quad (44)$$

$$E_c = 2m_{pl} c^2 \left(\frac{m}{m_{pl}} \right)^2 \left(\frac{\ell_{pl}}{\ell} \right) \frac{A_2}{B_2} \quad (45)$$

$$E_s = 2m_{pl} c^2 \left(\frac{m}{m_{pl}} \right)^5 \frac{A_3}{(B_3)^2} \quad (46)$$

where ℓ_{pl} is the Planck length and ℓ is the dimension of the box (or cylinder) in which the particle is confined. From the perspective of state-vector reduction, a particle prepared in a plane-wave or cylindrical state will have a radically different time-scale of collapse than the same particle prepared in a spherically symmetric state. A neutron in a spherically symmetric state will have a collapse time of $\Delta t \sim 10^{53}s$ whereas a neutron confined to a square box 10m in cross-sectional size will have a collapse time of only $\Delta t \sim 10^{22}s$. Both of these times are much longer than the age of the universe. However by confining a neutron to a rectangular pipe whose width is on the order of 10 picometers, the collapse time becomes on the order of 10^6s , or about 12 days. Such an experiment would push the limits of current technology.

Note that for $D = 4$ the probability is scale-invariant, and it is not possible to solve (16) for λ . The energy scale in this dimension is independent of the wavefunction normalization. If we work through the problem using the \mathfrak{K} rescaled equations (18) and (19), we can recover an expression for the energy eigenvalues in this dimension using (38). When rescaling equations (32) we can choose to absorb \mathfrak{K} in the rescaling of the expansion coefficients: $(\mathcal{A}, \mathcal{B}) \rightarrow (\mathfrak{K}\lambda^2 A, \mathfrak{K}B)$ where A and B are equivalent to (39) and (40). Then solving (19) for \mathfrak{K} we get

$$\mathfrak{K} = \frac{\Gamma(2)}{2\pi^2 \alpha^2 B_4} \quad (47)$$

We use this normalization to solve for the energy eigenvalue when $D = 4$

$$E_4 = \beta \mathcal{A}_4 = \beta \mathfrak{K} \lambda_4^2 A_4 = \lambda_4^2 \beta \left(\frac{\Gamma(2)}{2\pi^2 \alpha^2} \right) \frac{A_4}{B_4} \quad (48)$$

Clearly we are left with an unsolved parameter, λ_4 , which is at this point completely arbitrary. This is a consequence of the introduction of the rescaling constant \mathfrak{K} in the $D = 4$ case.

A graph of the invariant $A^{(4-D)}/B^2$ is instructive because it reflects the behaviour of the energy eigenvalue solutions with bound state that is independent of the value of the D -dimensional Newton constant. Figure 10 shows that the energy eigenvalues are positive and increase with bound state for $D < 3$, consistent with what we expect for the form of the classical Newtonian gravitational potential in these dimensionalities. Figure 11 shows that the energy eigenvalues are negative for dimensions greater than 3. In this graph, the values of λ_4 and r_0 are set to unity for $D = 4$.

4 Conclusions

We have presented the preliminary results of our numerical and analytical investigation of the Schroedinger-Newton equations in D dimensions under the assumption of spherical symmetry. Consistent with previous analyses in 3 dimensions, we have demonstrated numerically the existence of a discrete set of “bound-state” solutions for the equations, which are associated with different energy eigenvalues. The bound states appear to accumulate for dimensions greater than or equal to six. The bound state energies for $D < 3$ are positive and increase with increasing bound state number. For $D \geq 3$ the energies are negative and converge to zero with increasing eigenvalue number. This is consistent with Moroz [4], who found that for $D = 3$ the energy eigenvalues converge to zero as the order of the bound state increases. For higher dimensions, the variation of the eigenvalues with bound state becomes less pronounced, as illustrated in Figure 11. For $D > 5$ we find that the eigenvalues accumulate, and we are unable to determine any energy differences between bound states to 10 significant digits.

Moroz ventured further in the three dimensional study to assign numerical estimations for timescales of collapse, using the lowest bound state solutions to determine energy eigenvalues. However the analogous calculation using general D dimensions is not strictly correct due to possible dimensional variation in Newton’s constant G_D which appears in (9). We will not attempt to predict the variation of G_D with dimension here, although we note that Kaluza-Klein theory suggests that G_D is proportional to the three dimensional G_3 divided by the square root of the volume element of the extra space dimensions. We leave any such numerical estimates for future work.

However if we interpret the lower-dimensional cases to be systems in three spatial dimensions with either planar or cylindrical symmetry, a different picture emerges. The collapse time scale depends not only on the mass of the particle but also on the size of the box in which it is confined. If the gravitational influence of the box can be neglected and if the gravitational field of the neutron in a plane wave state has planar symmetry, then we find that the collapse timescales differ enormously between the two situations. A free neutron under its own gravitational self-influence will experience a collapse of its wavefunction on a timescale many orders of magnitude larger than the age of the universe, whereas one confined to a rectangular pipe whose cross-sectional width is about

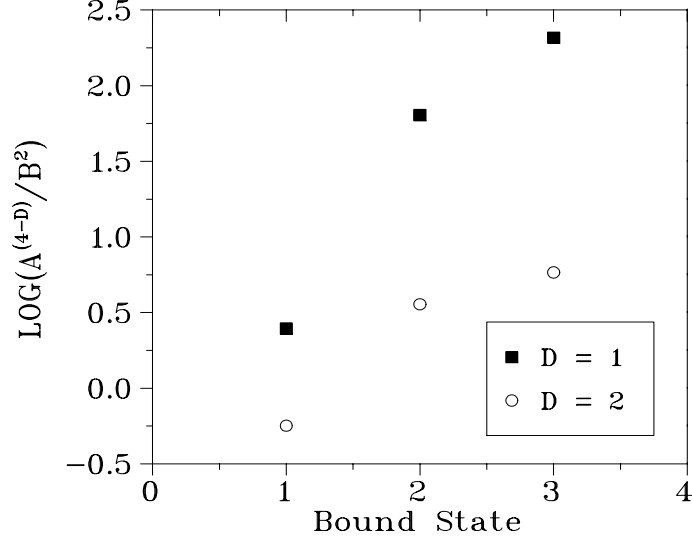


Figure 10: Behaviour of the invariant $A^{(4-D)}/B^2$ for the first three bound states for $D = 1$ and 2

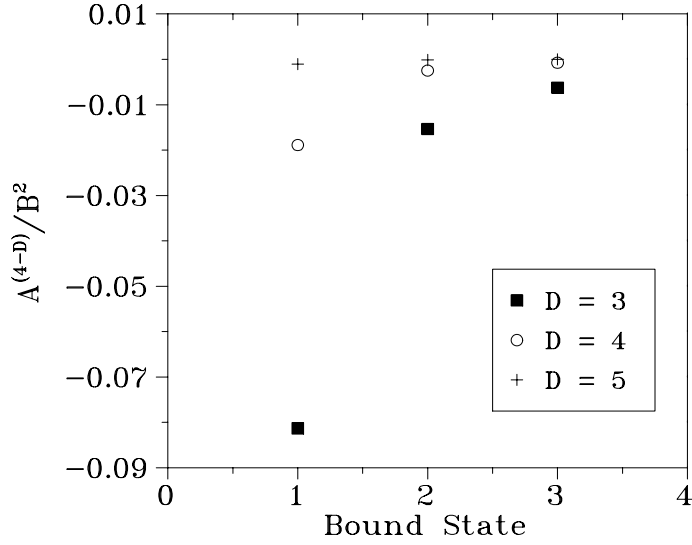


Figure 11: Behaviour of the invariant for the first three bound states for $D = 3$ to 5 . In the four dimensional case, the invariant is A_4/B_4 . The value of $A^{(4-D)}/B^2$ for $D = 6$ is equal to zero within the accuracy of our calculation

10 picometers in size will decay in about two weeks.

An obvious extension of our work involves introducing external field into our version of the SN equations, in the form of a point source perturbation. This can be added to the Schroedinger equation to give the perturbed system

$$-\frac{\hbar^2}{2m}\nabla^2\Psi + U\Psi - \frac{M}{r}\Psi = E\Psi \quad (49)$$

$$\nabla^2 U = 4\pi G m^2 |\Psi|^2 \quad (50)$$

Here M is the mass which generates the perturbing field. This system is quite interesting in the respect that the form of the bound state solutions is drastically altered from that of the “free field” equations considered in this paper. A preliminary study of this system indicates that each bound state solution seems to have three regions. For small M , the solutions asymptotically approach those of the free field or non-perturbed case, as expected. For moderate values of M , the equations are governed by both the free field and point source field, causing the values of S_0 , for which the solutions do not diverge, to alter considerably. Finally, for very large M , the solutions are governed almost entirely by the point source perturbation, and become similar to the hydrogen atom solutions of the Schroedinger equation. The effect of increasing dimension causes the “transition” values of M (which separate the different regions of the solutions) to decrease. A full study of the point source perturbation is currently in progress.

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